Linear Algebra-I
Semestral exam
Bmath First Year, AY: 24-25

Time: 3

Total marks: 50

Answer **any five** questions. No marks will be awarded in absence of proper justification.
Notations: *For a matrix A, $\mathcal{C}(A)$ denotes the column space of A, $\mathcal{R}(A)$ denotes the row space of A, and $\mathcal{N}(A)$ denotes the null space of A.*

(1) Let V and W be vector spaces over a field F, of dimensions n and m respectively. Let $\mathrm{Hom}_F(V,W)$ denote the set of all linear transformations from V to W.
(a) Show that $\mathrm{Hom}_F(V,W)$ is a vector space over F.
(b) Find a basis of $\mathrm{Hom}_F(V,W)$.
(c) Hence determine the dimension of $\mathrm{Hom}_F(V,W)$. (3+5+2)

(2) Let $x_1 = (2\ 0\ 1\ 3)^t$, $x_2 = (0\ 3\ 1\ 1)^t$, and $x_3 = (2\ \ -6\ \ -1\ 1)^t$.
(a) Find a basis of $S = \text{span of } \{x_1, x_2, x_3\}$.
(b) Find a complement T of S which contains the vector $u = (4\ 0\ 8\ 0)^t$.
(c) By modifying T, get a complement W of S which does not contain u and find the projection of u into S along W. (2+4+4)

(3) (a) Let $T : V \to V$ be a linear operator, and let A be the matrix of T with respect to the basis $\mathcal{B}$ of V. Show that the matrix of T with respect to a new basis $\mathcal{B}'$ is $P^{-1}AP$, where P is the change of basis matrix, ie. $\mathcal{B}' = \mathcal{B}\ P$.
(b) Show that a matrix M is an idempotent (ie. $M^2 = M$) if and only if there exists an invertible matrix Q such that $Q^{-1}MQ$ is a diagonal matrix of the form $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$. (5+5)

(4) (a) Define rank-factorization of a non-zero matrix.
(b) Find the rank factorization of the following matrix :

$$\begin{bmatrix} 2 & 4 & 1 & -1 \\ 3 & 6 & 0 & 1 \\ -1 & -2 & -2 & 3 \end{bmatrix}.$$

(c) Show that if (P, Q) is a rank-factorization of A then $\mathcal{C}(P) = \mathcal{C}(A)$, $\mathcal{R}(Q) = \mathcal{R}(A)$ and $\mathcal{N}(Q) = \mathcal{N}(A)$. (2+4+4)

(5) (a) Reduce the following matrix to its Hermite canonical form using row operations:

$$A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 2 & 5 & 0 \\ 3 & 6 & 0 & 5 \\ 4 & 8 & 1 & 2 \end{bmatrix}.$$

(b) Find a basis of the null space of A, using the Hermite form. Justify your answer.
(c) Determine the rank of A. (5+3+2)

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(6) (a) Find the LU decomposition of the following square matrix, if possible:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 6 \\ 2 & 4 & 8 \end{bmatrix}.$$

(b) Use the LU decomposition to find solution of the system of equation $AX = b$ where

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(5+5)

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